Example 6: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Solve the linear system $A\mathbf{x} = \mathbf{b}$ by row reducing the augmented matrix $[A|\mathbf{b}]$ to the augmented matrix $[I_2|\mathbf{x}]$ in reduced row echelon form.

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 4 & 1 \end{array}\right] \underset{R_2 := R_2 - 3R_1}{\sim} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -2 & -2 \end{array}\right]$$

$$\underset{R_2 := -\frac{1}{2}R_2}{\sim} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \end{array}\right] \underset{R_1 := R_1 - 2R_2}{\sim} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right]$$

$\vec{x}$

$x_1 = -1$

$x_2 = 1$

Matrix Inverse Calculation: Let A be a $n \times n$ matrix that is invertible. Consider the problem of finding the inverse A^{-1} with column vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ such that

$$\Rightarrow \quad AA^{-1} = I_n \quad A^{-1} = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n] \tag{5}$$

Since

$$AA^{-1} = A\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & \cdots & A\mathbf{v}_n \end{bmatrix}, \qquad I_n = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix} \tag{6}$$

we must have $A\mathbf{v}_i = \mathbf{e}_i$ for all i in $1, \ldots, n$.

1. We can find $\mathbf{v}_i$ by row reducing the augmented matrix $[A|\mathbf{e}_i]$ to the augmented matrix $[I_n|\mathbf{v}_i]$ in reduced row echelon form.

2. We can find all n columns of A^{-1} such that $AA^{-1} = I_n$ simultaneously by row reducing the augmented matrix $[A|I_n]$ to the augmented matrix $[I_n|A^{-1}]$ in reduced row echelon form.